Pf (SKETCH): Case 1 $B = \emptyset$

$$\begin{tikzcd}
H^l(M, \partial M) \arrow[r, "{[M,\partial M] \cap -}"] & H_{n-l}(M) \\
H^l_c(M \setminus \partial M) \arrow[u, "\cong"] \arrow[r] \arrow[ur, phantom, "G"] & H_{n-l}(M \setminus \partial M) \arrow[u, "\cong"']
\end{tikzcd}$$

This follows from everything we did today.

Case 2. General case; want to reduce to Case 1.
$\{A,B\}$ is an excisive couple, i.e. $H_*(M, \partial A) \cong H_*(B, \partial B)$, as we will now show.
We can find U_A, U_B collar nbhoods, $A \cap B \subseteq U_A \subseteq A$, $A \cap B \subseteq U_B \subseteq B$
for $\partial A = A \cap B = \partial B$.

$U := (B \cup U_A) \times [0,1) \subseteq \partial M \times [0,1)$

$V := (A \cup U_B) \times [0,1) \subseteq \partial M \times [0,1)$

$U \cup V = \partial M \times [0,1) \simeq \partial M$

$U \cap V \simeq A \cap B$, $U \simeq B$, $V \simeq A$ $\Rightarrow$ $\{A,B\}$ is excisive. ✓

$$\begin{tikzcd}
\dots \arrow[r] & H^l(M, \partial M) \arrow[r] \arrow[dd, "{[M,\partial M] \cap -}", "\cong"', "\text{Case 1}"'] & H^l(M, A) \arrow[r] \arrow[dd, "{[M,\partial M] \cap -}"'] & H^l(\partial M, A) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^{l+1}(M, \partial M) \arrow[r] \arrow[dd, "{[M,\partial M] \cap -}"'] & \dots \\
 & & & H^l(B, \partial B) \arrow[d, "{[B,\partial B] \cap -}"', "\cong \text{Case 1}"] & & \\
\dots \arrow[r] & H_{n-l}(M) \arrow[r] & H_{n-l}(M, B) \arrow[r, "\partial"] & H_{n-l-1}(B) \arrow[r] & H_{n-l-1}(M) \arrow[r] & \dots
\end{tikzcd}$$

here we use that $\{A,B\}$ is excisive

We have seen a similar diagram: painful but not difficult to see commutativity.

5-lemma $\rightarrow$ PL duality.

Alexander duality: $K \subseteq S^n$ compact, proper non-empty, locally contractible subset

$$\begin{tikzcd}
\Rightarrow \tilde{H}_l(S^n \setminus K; \mathbb{Z}) \arrow[r, phantom, "\cong"] \arrow[d, equal, "PD", "\text{for } l \neq 0"'] & \tilde{H}^{n-l-1}(K; \mathbb{Z}) \\
H^l_c(S^n \setminus K) &
\end{tikzcd}$$

The interesting thing is that only the $\overbrace{\text{shape of K}}$ matters, not how it is embedded into S^n.

More generally, we have $\tilde{H}_l(S^n \setminus K; \mathbb{Z}) \cong \check{H}^{n-l-1}(K; \mathbb{Z})$ for Čech cohomology.

18.06.2018

Hurewicz Theorem

Literature: Waldhausen, Bredon. Definitely not Hatcher.

We have seen: $\pi_n S^k = 0 \quad \forall n < k$

$\pi_1 S^1 \cong \mathbb{Z}, \quad \pi_n S^1 = 0 \quad \forall n > 1$

$\pi_n S^k = ?$ Open, relatet to e.g. classification of mfs.

Goal: $\pi_n S^n \cong \mathbb{Z} \quad \forall n \geq 1$

Furthermore, $\pi_3 S^2 \cong \mathbb{Z}, \quad \pi_4 S^2 \cong \mathbb{Z}/2$

Recall the definition of (relative) htp gps:

let $(X, A), (Y, B)$ be pairs of spaces, $A \subseteq X, B \subseteq Y$

$[(Y,B),(X,A)] = C((Y,B),(X,A))/\simeq$

where $f \simeq g \iff \exists H: Y \times I \longrightarrow X, \; H(-,0) = f, \; H(-,1) = g, \; H(B \times I) \subseteq A$

and $C(Y,X)$ is the set of cont. maps, $C((Y,B),(X,A)) = \{ f \in C(Y,X) \mid f(B) \subseteq A \}$

$\pi_n(X, x_0) := [(S^n, *), (X, x_0)] \cong [(I^n, \partial I^n), (X, x_0)] \cong [(D^n, \partial D^n), (X, x_0)]$

$n = 0$: just a pointed set

$n = 1$: a group which might be non-abelian

$n \geq 2$: abelian group

Now let $x_0 \in A \subseteq X, \quad * \in \partial D^n \subseteq D^n$

$$\pi_n(X, A, x_0) = [(D^n, \partial D^n, *), (X, A, x_0)] \qquad \forall n \geq 1$$

$$\cong [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)]$$

where $J^{n-1} = \{ (t_1, \dots, t_n) \in I^n \mid t_i \in \{0,1\}$ for some $1 \leq i \leq n-1$ or $t_n = 1 \} \subseteq \partial I^n$,

i.e. J^{n-1} is the union of all $(n-1)$ dimensional faces except for

$I^{n-1} = \{ (t_1, \dots, t_n) \in I^n \mid t_n = 0 \}$.

For $[f], [g] \in \pi_n(X, A, x_0)$ with $n \geq 2$

or $[f], [g] \in \pi_n(X, x_0)$ with $n \geq 1$:

$$(f+g)(t_1, \dots, t_n) := \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \dots, t_n) & 1/2 \leq t_1 \leq 1 \end{cases}$$

This gives a group structure, abelian for $\pi_n(X, x_0) \quad \forall n \geq 2$

and $\pi_n(X, A, x_0) \quad \forall n \geq 3$.

Theorem. (Absolute Hurewicz Theorem)

Let X be simply connected (i.e. path-connected and π_1-trivial), $n \geqq 2$ and $\pi_i(X, x_0) = 0 \ \forall i \leqq n-1$.

Then $\tilde{H}_i(X; \mathbb{Z}) = 0 \ \forall i \leq n-1$ and $\pi_n(X, x_0) \cong H_n(X; \mathbb{Z})$.

Theorem. (Poincaré, Topology I)

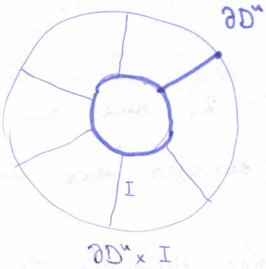
Let X be a path-connected space. Then there is a canonical homomorphism $h: \pi_1(X, x_0) \longrightarrow H_1(X; \mathbb{Z})$ such that the induced morphism $\pi_1(X, x_0)^{ab} \xrightarrow{\tilde{h}} H_1(X, \mathbb{Z})$ is an iso.

Action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$ $\forall n \geqq 1$:

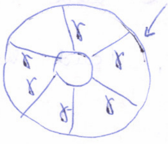
$[\gamma] \in \pi_1(A, x_0)$, $\gamma: (I, \partial I) \longrightarrow (A, x_0)$

$[f] \in \pi_n(X, A, x_0)$, $f: (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$

$$\begin{tikzcd}
\partial D^n \times 0 \underset{* \times 0}{\cup} * \times I \arrow[r, "f|_{\partial D^n} \cup \gamma"] \arrow[d, hook] & A \\
\partial D^n \times I \arrow[ur, dashed, "\exists \bar{f}"'] \arrow[ur, phantom, "G", shift left=4ex] &
\end{tikzcd}$$

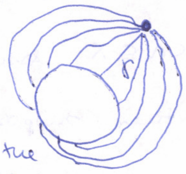


$\bar{f}$ exists by HEP:



note that this picture works in the absolute case only (see below)

this is the actual picture in the relative case



$$\begin{tikzcd}
D^n \times 0 \underset{\partial D^n \times 0}{\cup} \partial D^n \times I \arrow[r, "f \cup \bar{f}"] \arrow[d, hook] & X \\
D^n \times I \arrow[ur, dashed, "\exists F"'] \arrow[ur, phantom, "G", shift left=4ex] &
\end{tikzcd}$$

F exists again by HEP.

Note that $F|_{* \times I} = \gamma$.

Then $\pi_1(A, x_0) \curvearrowright \pi_n(X, A, x_0)$ is defined by

$[f] \circ [\gamma] := [F|_{D^n \times 1}] \in \pi_n(X, A, x_0)$

We have seen in Topology I that this is really an action,

$[\gamma] \circ (-): \pi_n(X, A, x_0) \xrightarrow{\cong} \pi_n(X, A, x_0)$ $\forall n \geqslant 2$ we have a group automorphism.

(The action in the absolute case is much easier: it is defined by

$$\begin{tikzcd}
S^n \times 0 \cup * \times I \arrow[r] \arrow[d, hook] & X \\
S^n \times I \arrow[ur, dashed] &
\end{tikzcd}$$

and restricting the dashed arrow to $S^n \times 1$.)

Recall the long exact sequence on htp gps:

$$\pi_n(A,x_0) \longrightarrow \pi_n(X,x_0) \longrightarrow \pi_n(X,A,x_0) \longrightarrow$$

$$\to \pi_{n-1}(A,x_0) \longrightarrow \pi_{n-1}(X,x_0) \longrightarrow \pi_{n-1}(X,A,x_0) \longrightarrow \dots$$

$$\to \pi_1(A,x_0) \longrightarrow \pi_1(X,x_0) \longrightarrow \pi_1(X,A,x_0) \longrightarrow$$

$$\to \pi_0(A,x_0) \longrightarrow \pi_0(X,x_0)$$

If A and X are path-connected and $\pi_1(A,x_0) \xrightarrow{\cong} \pi_1(X,x_0)$ then $\pi_1(X,A,x_0)=0$.

$[f]\in \pi_n(X,A,x_0)$, $f:(D^n,\partial D^n,*) \longrightarrow (X,A,x_0)$. $\partial[f] = [f|_{\partial D^n}]$

We have $\pi_1(A,x_0) \curvearrowright \pi_n(X,A,x_0)$. Suppose $n \geqq 2$.

$$\tilde{\pi}_n(X,A,x_0) := \pi_n(X,A,x_0)^{ab} \Big/ \langle [f]-[\gamma][f] \rangle \ \forall [f]\in\pi_n(X,A) \ \forall \gamma\in\pi_1(A)$$

Rmk. For $n=2$, one can drop the abelianisation from the definition if $\langle - \rangle$ denotes the normal subgp generated. (Exc.)

Thm. (Relative Hurewicz Theorem) Suppose $n\geqq 2$ and X, A are path-connected spaces, $x_0 \in A \subseteq X$. Assume $\pi_1(A,x_0) \xrightarrow{\cong} \pi_1(X,x_0)$, and hence $\pi_1(X,A,x_0)=0$.
Further suppose $\pi_i(X,A,x_0)=0$ $\forall i \leq n-1$.
Then $H_i(X,A;\mathbb{Z})=0$ $\forall i\leq n-1$ and $\tilde{\pi}_n(X,A,x_0) \cong H_n(X,A;\mathbb{Z})$.

Define maps

$h: \pi_n(X,x_0) \longrightarrow H_n(X;\mathbb{Z})$ and $h: \pi_n(X,A,x_0) \longrightarrow H_n(X,A;\mathbb{Z})$.

We need to fix orientations for spheres and discs:

$H_n(D^n,\partial D^n;\mathbb{Z}) \cong \mathbb{Z}$. has no canonical choice for generator (a priori).

$H_n(\Delta^n,\partial\Delta^n;\mathbb{Z})\cong\mathbb{Z}$ does have a preferred generator though:

$[\mathrm{id}: \Delta^n \longrightarrow \Delta^n] \in H_n(\Delta^n, \partial D^n;\mathbb{Z})$ is this generator.

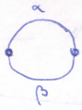
Once and for all fix a homeo $D^n \cong \Delta^n$ (and $\partial D^n \cong \partial \Delta^n$);

also fix $D^n/_{\partial D^n} \cong S^n$.

With these homeos fixed, we obtain a preferred generator

$\iota_n \in H_n(D^n, S^{n-1};\mathbb{Z})$ and $\iota_n\in H_n(S^n;\mathbb{Z})$.

Recall that $[\alpha]-[\beta]$ generates $H_1(S^1;\mathbb{Z})$ where



and this yields a generator of $H_n(S^n;\mathbb{Z})$ via suspension. One can show that this construction agrees with the one we just introduced (in some way ...?)

Define a map $h: \pi_n(X,A,x_0) \longrightarrow H_n(X,A;\mathbb{Z})$, called the (relative) Hurewicz homomorphism

$$[f] \longmapsto f_*(\iota_n)$$

Here $f: (D^n,\partial D^n,*) \longrightarrow (X,A,x_0)$; forgetting about base pts this yields $(D^n,S^{n-1}) \xrightarrow{f} (X,A)$.

This induces $f_*: H_n(D^n,S^{n-1}) \longrightarrow H_n(X,A)$

$$\iota_n \longmapsto f_*(\iota_n)$$

Homotopy invariance of homology $\Rightarrow$ h is well-defined.

The absolute Hurewicz map is defined as

$$h: \pi_n(X,x_0) \longrightarrow H_n(X;\mathbb{Z})$$
$$[f] \longmapsto f_*(\iota_n)$$

Here $f: S^n \longrightarrow X$, $\quad f_*: H_n(S^n;\mathbb{Z}) \longrightarrow H_n(X;\mathbb{Z})$

$$\iota_n \longmapsto f_*(\iota_n)$$

Again by htp invariance we have that h is well-defined.

The statements that these are homomorphisms will be checked later or in the exercises. Oh wait, we are doing the absolute case right now.

Prop. h is a homomorphism in the absolute case.

Pf: $[f],[g] \in H_n(X,x_0)$. Then $[f]+[g]$ is represented by

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f\vee g} X, \qquad [f]+[g] = [(f\vee g)\circ \text{pinch}]$$

We have the following maps: $S^n \xrightarrow{\text{pinch}} S^n \vee S^n \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} S^n$

$pr_1 \circ \text{pinch} \simeq id, \quad pr_2 \circ \text{pinch} \simeq id$

$$\begin{tikzcd}
H_n(S^n;\mathbb{Z}) \arrow[r, "\text{pinch}_*"] \arrow[dr, "\Delta", "\text{diagonal}"'] & H_n(S^n\vee S^n;\mathbb{Z}) \arrow[r, "(f\vee g)_*"] \arrow[d, "\cong"] & H_n(X;\mathbb{Z}) \\
 & H_n(S^n;\mathbb{Z}) \oplus H_n(S^n;\mathbb{Z}) \arrow[ur, "f_*\oplus g_*"'] &
\end{tikzcd}$$

$$h([f]+[g]) = ((f\vee g)\circ \text{pinch})_*(\iota_n) = (f\vee g)_*((\text{pinch})_*(\iota_n))$$
$$= (f_*\oplus g_*)\circ\Delta(\iota_n) = (f_*\oplus g_*)(\iota_n,\iota_n)$$
$$= f_*(\iota_n)+g_*(\iota_n) = h[f]+h[g].$$

$$\begin{tikzcd} \pi_n S^n \arrow[r, "h"] \arrow[rr, bend right] & H_n S^n \arrow[r, "\cong"] & \mathbb{Z} \end{tikzcd}$$

$f: S^n \longrightarrow S^n$

$[f] \longmapsto \deg f$

$$\begin{tikzcd} z_n \in H_n S^n \arrow[r, "f_*"] \arrow[d, leftrightarrow, "\cong"] & H_n S^n \arrow[d, phantom, "\cong"] \\ 1 \in \mathbb{Z} \arrow[r, "\deg f"] & \mathbb{Z} \end{tikzcd}$$

$f_*(z_n) = \deg(f) \cdot z_n$ "=" $\deg(f) \cdot 1$

Consequently: $h: \pi_n S^n \longrightarrow H_n S^n$ is surjective since $\deg(\mathrm{id}) = 1$.

$\Rightarrow \pi_n S^n$ contains $\mathbb{Z}$ as a direct summand.

Today every (co)homology group/ring is over $\mathbb{Z}$. 20.06.2018

Lemma. $n \geq 1$ and assume that $\pi_{n-1} S^{n-1} \cong \mathbb{Z}$ if $n \geq 2$.
Then $f: (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ has degree 1 iff f is homotopic to the identity, has degree -1 iff it is homotopic to the reflection $(x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, -x_n)$.

Note. We already know the assumption for $n=2$ since $\pi_1 S^1 \cong \mathbb{Z}$. For larger n we just treat this as a black box.

Pf: $n=1$: $f: (D^1, \partial D^1) \longrightarrow (D^1, \partial D^1)$

Suppose that $\deg(f) = \pm 1 \Rightarrow f(\partial D^1) = \partial D^1$ because ↯ otherwise $f(\partial D^1)$ is just one point, and since $\pi_1(D^1) = 0$ this yields $f \simeq_{\mathrm{rel}\, \partial D^1} \mathrm{const}$, hence $\deg(f) = 0$. ↯

If $f(-1) = -1$, $f(1) = 1 \Rightarrow f \simeq_{\mathrm{rel}\, \partial D^1} \mathrm{id}$ by linear homotopy

If $f(-1) = 1$, $f(1) = -1 \Rightarrow f \simeq_{\mathrm{rel}\, \partial D^1} -\mathrm{id}$, i.e. the reflection. ✓

$n \geq 2$: $f, g: (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$

Claim. $f \simeq_{\mathrm{rel}\, \partial D^n} g$ iff $f|_{\partial D^n} \sim g|_{\partial D^n}$.

Pf: "$\Rightarrow$" is obvious.

Suppose $f|_{\partial D^n} \sim g|_{\partial D^n}$, let $H: \partial D^n \times I \longrightarrow \partial D^n$ be the homotopy.

$$\begin{tikzcd} \partial D^n \times I \cup D^n \times 0 \cup D^n \times 1 \arrow[r, "H \cup f \cup g"] \arrow[d, hook] & D^n \simeq * \\ D^n \times I \arrow[ur, dashed, "\exists \bar{H} \text{ extension}"'] & \end{tikzcd}$$

$\bar{H}|_{\partial D^n \times I} = H$, hence takes values in ∂D^n $\Rightarrow$ $\bar{H}$ is a htp. b/w f, g. □

Hence $f : (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ is a htp equivalence iff $f|_{\partial D^n}$ is one.

Moreover, $f \sim id$ iff $f|_{\partial D^n} \simeq id$,

$f \sim$ reflection iff $f|_{\partial D^n} \simeq$ reflection.

These follow immediately from the Claim.

By the naturality of the connecting homomorphism:

$$\begin{tikzcd} H_n(D^n, \partial D^n) \arrow[r, "\partial", "\cong"'] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(\partial D^n) \arrow[d, "(f|_{\partial D^n})_*"] \\ H_n(D^n, \partial D^n) \arrow[r, "\partial", "\cong"'] & H_n(\partial D^n) \end{tikzcd}$$

because $D^n \simeq *$ and LES.

$\deg f = 1 \Rightarrow \deg f|_{\partial D^n} = 1.$

$\deg f = -1 \Rightarrow \deg f|_{\partial D^n} = -1$ since $\deg f = \deg f|_{\partial D^n}$.

Hurewicz:

$$\begin{tikzcd} h : \pi_n S^{n-1} \arrow[r, two heads] \arrow[rr, bend right, "\deg"'] & H_{n-1}(S^{n-1}) & \cong \mathbb{Z} \end{tikzcd}$$

assumption $\to$ $\pi_n S^{n-1} \cong \mathbb{Z}$

h is surjective $\Rightarrow$ h is an iso.

$$\Rightarrow \begin{tikzcd} {[S^{n-1}, S^{n-1}]} \arrow[r, two heads, "H_{n-1}", "\cong"'] \arrow[rr, bend right, "\deg"'] & \mathrm{Hom}_{\mathbb{Z}}(H_{n-1}S^{n-1}, H_{n-1}S^{n-1}) & \cong \mathbb{Z} \end{tikzcd}$$

deg surjective $\Rightarrow$ so is H_{n-1}.

If $f \simeq f'$ s.t. $f'(*) = *$, i.e. f' preserves base pt, $\bar{f} \simeq \bar{f}'$, $\bar{f}'(*) = x$.

$f : S^{n-1} \longrightarrow S^{n-1}$, $f(*) = x$, $\gamma : [0,1] \to S^{n-1}$, $\gamma(1) = *$, $\gamma(0) = x$

$$\begin{tikzcd} S^{n-1} \times 0 \cup * \times [0,1] \arrow[r, "f \cup \gamma"] \arrow[d, hook] & S^{n-1} \\ S^{n-1} \times I \arrow[ur, "H"'] & \end{tikzcd}$$

$H(-, 1) = f'$

$H_{n-1}(f) = H_{n-1}(\bar{f}) \Rightarrow H_{n-1}(\bar{f}') = H_{n-1}(f') \Rightarrow \bar{f}' \simeq f' \Rightarrow f \simeq \bar{f}$.

"Everything I'm saying today is basically trivial. You probably appreciate that we are doing stuff that's understandable. That ends this week."

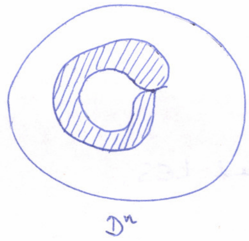
$\deg(f) = 1 \Leftrightarrow \deg(f|_{\partial D^n}) = 1 \Leftrightarrow f|_{\partial D^n} \simeq id \underset{\text{Claim}}{\Leftrightarrow} f \simeq id$

Similarly one can show $\deg(f) - 1 \Leftrightarrow f \simeq$ reflection. □

Local degree

$f: D^n \longrightarrow D^n$ cont. map s.t. $f|_{\mathring{D}^n} : \mathring{D}^n \xrightarrow{\cong} f(\mathring{D}^n)$

We define $\operatorname{ldeg}(f) \in \mathbb{Z}$ below.



Take any $x \in \mathring{D}^n$.

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\vdots", "\cong"'] \arrow[d, dashed, "\exists"'] & H_n(D^n, D^n \setminus x) & H_n(\mathring{D}^n, \mathring{D}^n \setminus x) \arrow[l, "exc"', "\cong"] \arrow[d, "f_*", "\cong"'] \\
H_n(D^n, \partial D^n) \arrow[r, "\cong"'] & H_n(D^n, D^n \setminus f(x)) & H_n(f(\mathring{D}^n), f(\mathring{D}^n) \setminus f(x)) \arrow[l, "exc"', "\cong"]
\end{tikzcd}$$

Under the dashed arrow, z_n gets sent to $d \cdot z_n$.

Let $\operatorname{ldeg}(f) := d$. This is independent of the choice of x:

the idea is that for $x, x' \in \mathring{D}^n$ there is a ball $B \ni x, x'$.

Exercise: one can go through $H_n(\mathring{D}^n, \mathring{D}^n \setminus B)$.

Homotopy addition theorem

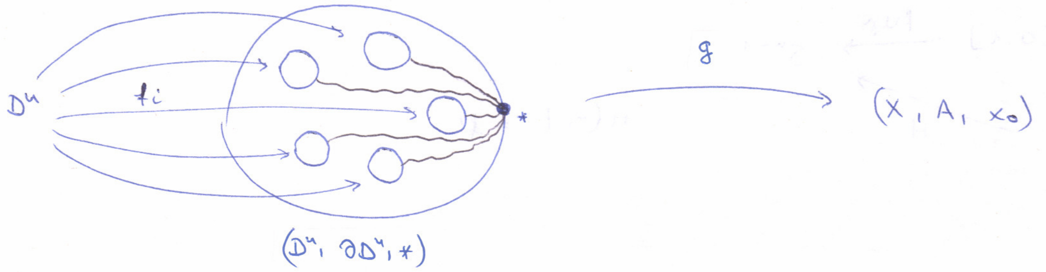
$\pi_1(A, x_0) \curvearrowright \pi_n(X, A, x_0)$ where $x_0 \in A \subseteq X$

Given a path $\gamma: I \longrightarrow A$, $\gamma(0) = x_0$, $\gamma(1) = x$

this induces a map $[\gamma]_* = \gamma_* : \pi_n(X, A, x_0) \xrightarrow{\cong} \pi_n(X, A, x_0)$

This depends only on the htp class of γ rel ∂I.

γ_* is an iso since the reverse path induces an inverse to it.



Topology II, lecture 19-20

$$D^n \xrightarrow{\quad f_i \quad} D^n \qquad i=1,\dots,k; \qquad n\geq 2$$

$$f_i|_{\mathring{D}^n} : \mathring{D}^n \xrightarrow{\cong} f(\mathring{D}^n) \quad \text{open}$$

$f_i(\mathring{D}^n) \subseteq D^n$ are all disjoint

Suppose we are given a map $g : (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$

Assume $g\left(D^n \setminus \bigcup_{i=1}^{k} f(\mathring{D}^n)\right) \subseteq A$.

Consider the composites $g \circ f_i : D^n \longrightarrow X$, then we have that $g\circ f_i(\partial D^n) \subseteq A$, but $g\circ f_i(*)$ need not be x_0.

Choose paths ω_i from $f_i(*)$ to $*$. such that $\operatorname{Im}\omega \subseteq D^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{D}^n)$

This gives paths $\gamma_i := g\circ\omega_i$ from $g\circ f_i(*)$ to x_0, since $g(*) = x_0$.

<u>Theorem.</u> (Homotopy addition theorem)

Let $x_0 \in A \subseteq X$, A and X path-connected spaces.

Suppose we are given $f_i : D^n \longrightarrow D^n$ $i=1,\dots,k$ such that $f_i(\mathring{D}^n)$ are open and $f_i|_{\mathring{D}^n} : \mathring{D}^n \xrightarrow{\cong} f_i(\mathring{D}^n)$. Further assume that all $f_i(\mathring{D}^n)$ are disjoint. Given a $g : (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$ (i.e. $[g] \in \pi_n(X, A, x_0)$) such that $g\left(\mathring{D}^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{D}^n)\right) \subseteq A$ we have an identity

$$[g] = \sum_{i=1}^{k} \operatorname{ldeg}(f_i)\cdot(\gamma_i)_*[g\circ f_i] \in \tilde{\pi}_n(X, A, x_0). \qquad (\#)$$

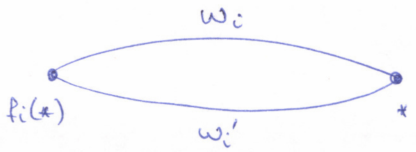
Note that there is a more general version with π^{ab} instead of $\tilde{\pi}$.

<u>Rmk.</u> $\operatorname{ldeg}(f_i) \in \{\pm 1\}$ $\forall i$ since $f_i|_{\mathring{D}^n}$ are homeomorphisms.

$(\gamma_i)_*[g\circ f_i]$ are independent of the choice of the ω_i.

Pf. Choose another path ω_i' from $f_i(*)$ to $*$, $\gamma_i' := g\circ\omega_i'$ a path from $g\circ f_i(*)$ to $g(*) = x_0$. WTS: $(\gamma_i)_*[g\circ f_i] = (\gamma_i')_*[g\circ f_i]$

Take a concatenation of paths $\omega := \omega_i' * \omega_i^{-1}$, $\gamma := g\circ\omega = \gamma_i' * \gamma_i^{-1}$



WTS $\gamma_*\big((\gamma_i)_*[g\circ f_i]\big) = (\gamma_i)_*[g\circ f_i]$

This will follow if we show that γ_* acts trivially. Note that $[\gamma]\in\pi_1(A,x_0)$ by def.

$$\begin{tikzcd}
{} & {[\omega]\in\pi_1\big(D^n\setminus\bigcup_{i=1}^{k} f_i(\mathring{D}^n),\ *\big)} \arrow[r, "g_*"] & {\pi_1(X,x_0)\ni g_*[\omega]=[\gamma]=1} \\
{} \arrow[u, maps to] & {\pi_2\big(D^n,\ D^n\setminus\bigcup_{i=1}^{k} f_i(\mathring{D}^n),\ *\big)} \arrow[u, "\partial"', "\cong"] \arrow[r, "g_*"] \arrow[ur, phantom, "G"] \arrow[r, bend right=30] & {\pi_2(X,A,x_0)\ni[\lambda]} \arrow[u, "\partial"']
\end{tikzcd}$$

If $n\geq 3$: $\pi_1\big(D^n\setminus\bigcup_{i=1}^{k} f_i(\mathring{D}^n)\big)=0 \Rightarrow$ ✓

$n=2$: $g_*[\omega] = \partial[\lambda] \Rightarrow \partial[\lambda]=[\gamma]$

$\partial[\lambda]\cdot a = \gamma_*(a)$ $\qquad\forall a\in\pi_2(X,A,x_0)$

$\partial[\lambda]\cdot a = \underbrace{[\lambda]\cdot[a]\cdot[\lambda^{-1}]}$ $\qquad$ (Lemma of Whitehead)

$= [a]$ in $\tilde{\pi}_2(X,A,x_0)$ $\qquad\boxed{R}$

Now the formulation of HAT is well-defined.

25.06.2018

Lemma 1. We can choose any path γ_i in A from $gf_i(*)$ to x_0, i.e. (#) is independent of the choice of the γ_i's.

Pf: γ_i and γ_i' paths from $gf_i(*)$ to x_0.

WTS $\gamma_{i*}[g\circ f_i] = \gamma'_{i*}[g\circ f_i]$ $\qquad$ (Slight AON: should write $[\gamma_i]_*[g\circ f_i]$)

$$\begin{tikzcd}
gf_i(*) \arrow[r, squiggly, shift left, "\gamma_i"] \arrow[r, squiggly, shift right, "\gamma_i'"'] & x_0
\end{tikzcd}$$

$$\underbrace{(\gamma_i'\gamma_i^{-1})}_{\in\pi_1(A,x_0)}{}_*\big(\gamma_{i*}[g\circ f_i]\big) = \gamma'_{i*}\gamma^{-1}_{i*}\gamma^*_i[g\circ f_i] = \gamma'_{i*}[g\circ f_i]$$

$\boxed{1}$

Lemma 2. (Replacing g) Given $g, g' : (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$ homotopic as maps of pairs (i.e. the homotopy need not respect the base points) then $[g]=[g']$ in $\tilde{\pi}_n(X,A,x_0)$.

Pf: $H: D^n\times I \longrightarrow X$ homotopy b/w g and g', $H(\partial D^n\times I)\subseteq A$.

$\gamma := H(*, -)$ $(*\in\partial D^n)$ $\Rightarrow$ γ is in $\pi_1(A,x_0)$ and $\gamma_*[g]=[g']$. $\qquad\boxed{2}$

Topology II, lecture 20

Lemma 3. (Replacing f_i) Given $g : (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$ and $f, f' : D^n \longrightarrow D^n$ and a homotopy H connecting f and f' s.t. $H(\partial D^n \times I) \subseteq g^{-1}(A)$, we have $\gamma_* [g \circ f] = \gamma'_* [g \circ f']$ in $\tilde{\pi}(X, A, x_0)$ where $g \circ f(*) \overset{\gamma}{\rightsquigarrow} x_0 \leftsquigarrow g \circ f'(*)$.

Pf: $\lambda := gH(*, -)$ path, $\lambda : I \longrightarrow A$.

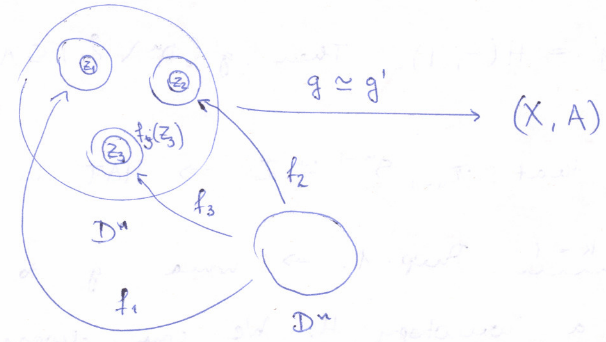
$gH(\partial D^n \times I) \subseteq A \Rightarrow \lambda_* [g \circ f] = [g \circ f']$

$$\gamma'_* [g \circ f'] = \gamma'_* \lambda_* [g \circ f] = \underbrace{(\gamma' \lambda \gamma^{-1})_*}_{\in \pi_1(A, x_0)} (\gamma_* [g \circ f])$$

We will use these lemmata numerous times throughout the pf. of HAT.

Prop. 1. Suppose (X, A, x_0), g, f_i $(i = 1, \dots, k)$ are as in the formulation of HAT. Further suppose $Z_i \subseteq f_i(\mathring{D}^n)$ is a closed ball (i.e. anything homeomorphic to a closed ball). Then $\exists g' : (D^n, \partial D^n, *) \to (X, A, x_0)$ such that $g'(D^n \setminus \bigcup_{i=1}^k \mathring{Z}_i) \subseteq A$ and $\exists H$ homotopy b/w g and g' s.t. $H|_{(D^n \setminus \bigcup_{i=1}^k f_i(\mathring{D}^n)) \times I} = g \circ pr_1$ (in particular, this lives in A).

Pf: It suffices to show that for fixed i there exist g' and H s.t. $H : g \simeq g'$ and $g'(D^n \setminus \mathring{Z}_i) \subseteq A$ and $H|_{(D^n \setminus f_i(\mathring{D}^n)) \times I} = g \circ pr_1$



Wlog wlma $f_i(0) \in Z_i$: otherwise choose a homeo $\varphi : D^n \xrightarrow{\cong} D^n$ with $\varphi(0) = z_0$ such that $z_0 \in \mathring{Z}_i$ and $\varphi|_{\partial D^n} = id$ (such a φ exists).

$\Rightarrow \varphi \simeq id$ rel ∂. Lemma 3 $\Rightarrow$ $f_i \circ \varphi$ and f_i are homotopic in the appropriate sense.

Now choose $0 \in Q_i \subseteq D^n$ concentric ball (around 0) s.t. $f_i(Q_i) \subseteq \mathring{Z}_i$ and consider the pushout

$$\begin{tikzcd}
S^{n-1} = \partial D^n \arrow[r, "f_i|_{\partial D^n}"] \arrow[d, hook] & D^n \setminus f_i(\mathring{D}^n) \arrow[d] \arrow[rdd, hook, bend left] & \\
D^n \arrow[r] \arrow[rrd, bend right, "f_i"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rd, dashed, "\exists!\ \cong"] & \\
 & & D^n
\end{tikzcd}$$

The dashed arrow is a homeo b/c it is a bijection of cpt T2-spaces.

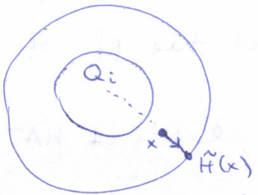
Since cartesian products preserve pushouts, we get another pushout

$$\begin{tikzcd}
\partial D^n \times I \arrow[r, "f|_{\partial D^n} \times 1"] \arrow[d, hook] & (D^n \setminus f_i(\mathring{D}^n)) \times I \arrow[d] \arrow[dr, hook] \arrow[ddrr, bend left, "g \circ pr_1"] & & \\
D^n \times I \arrow[r] \arrow[drr, bend right, "f_i \times 1"'] \arrow[ddrrr, bend right, "g \circ f_i \circ \tilde{H}"'] & P \times I \arrow[dr, "\cong"] \arrow[ddrr, bend left, "H'"] & & \\
& & D^n \times I \arrow[dr, "H"] & \\
& & & X
\end{tikzcd}$$

Goal: define some $H: D^n \times I \longrightarrow X$.

Instead we define $H': P \times I \longrightarrow X$.

$\tilde{H}: D^n \times I \longrightarrow D^n$ where $\tilde{H}(-,0) = id$, $\tilde{H}(x,1) = \begin{cases} \text{radial projection to } \partial & x \notin \mathring{Q}_i \\ \text{radial homeo applied to } x & x \in Q_i \end{cases}$



radial proj to ∂

radial homeo

$\tilde{H}\Big|_{\partial D^n \times I} = id$

Univ. prop. of pushouts $\Rightarrow$ $\exists H': P \times I \longrightarrow X$ $\Rightarrow$ we get $H: D^n \times I \longrightarrow X$

Set $g' := H(-,1)$. Then $g'(D^n \setminus \mathring{Z}_i) \subseteq A$, proving the Prop. $\boxed{1}$

Claim. $\pi_{n-1} S^{n-1} \cong \mathbb{Z}$ $\Rightarrow$ HAT in dimension n.

Case $k=1$. Prop. 1. $\Rightarrow$ wma g to be replaced by $g': (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$ with a homotopy H. We can choose $Z_1 \subseteq f_1(\mathring{D}^n)$ to be a concentric ball. (If $0 \notin f_i(\mathring{D}^n)$ then once again take a homeomorphism $\varphi: D^n \longrightarrow D^n$, $\varphi|_{\partial D^n} = id$, $\varphi(z_1) = 0$, $z_1 \in \mathring{Z}_1$. Then we can work with $g \circ \varphi^{-1}$ and $\varphi \circ f_1$ instead of g and f; these are equivalent elts in $\tilde{\pi}_n$.

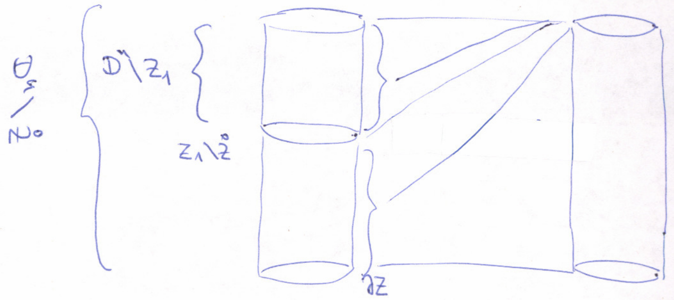
$g'(D^n \setminus \mathring{Z}_i) \subseteq A$ and $H\Big|_{D^n \setminus f_1(\mathring{D}^n) \times I}$ is constant

$[g] = [g']$ in $\tilde{\pi}_n(X, A, x_0)$ and by Lemma 2: $\gamma_{1*}[g \circ f] = \gamma_{1*}[g' \circ f]$.

NTS $ldeg(f_1)\, \gamma_{1*}[g' \circ f_1] = [g']$.

Consider $D^n \setminus \mathring{Z}$ where $Z \subseteq \mathring{Z}_1$ is a smaller concentric ball around 0.

Think of $D^n \setminus \mathring{Z}$ as a cylinder:



There exists $p : D^n \setminus \mathring{Z} \longrightarrow D^n \setminus \mathring{Z}$

such that $p|_{Z_1 \setminus \mathring{Z}} : Z_1 \setminus \mathring{Z} \xrightarrow{\cong} D^n \setminus \mathring{Z}$,

$p|_{\partial Z} = id$, $p|_{\partial D^n} = id$.

This p is homotopic to id via a homotopy $P : D^n \setminus \mathring{Z} \longrightarrow D^n \setminus \mathring{Z}$

with $P|_{\partial Z \times I} = id$, $P(-,0) = p$, $P(-,1) = id$, $P(D^n \setminus \mathring{Z}_1 \times I) \subseteq D^n \setminus \mathring{Z}_1$,

$P|_{\partial D^n \times I} = pr_1$.

Now extend p and P to the whole D^n by taking id of $\mathring{Z}$.

$q : D^n \longrightarrow D^n$, $Q : D^n \times I \longrightarrow D^n$, $q|_{D^n \setminus \mathring{Z}} = p$, $q|_{\mathring{Z}} = id$, $Q|_{D^n \setminus \mathring{Z} \times I} = P$.

Set $f_1' := q \circ f_1$. Then $f_1'(\partial D^n) \subseteq \partial D^n$

$\overline{Q} := Q \circ (f_1 \times id)$ Then $\overline{Q} : f_1' \simeq f_1$ since $q \simeq id$.

Lemma 3: $\gamma_*[g' \circ f_1] = \gamma_*[g' \circ f_1']$

<u>Sublemma.</u> $\operatorname{ldeg}(f_1) = \deg(f_1')$. (To be proven later.)

If $\operatorname{ldeg}(f_1) = 1 \Rightarrow \deg(f_1') = 1 \Rightarrow f_1' \underset{rel\,\partial}{\simeq} id \Rightarrow [g'] \overset{L3}{=} \gamma_{1*}[g \circ f_1'] = \gamma_{1*}[g \circ f]$.

If $\operatorname{ldeg}(f_1) = -1 \Rightarrow \deg(f_1') = -1 \underset{\pi_{n-1}S^{n-1} \cong \mathbb{Z}\ ^{*}}{\Rightarrow} f_1' \underset{rel\,\partial}{\simeq}$ reflection $\Rightarrow -[g'] = \gamma_*[g' \circ f_1']$

This proves the $k=1$ case of the Claim modulo the Sublemma.

Pf OF SL: $x \in \mathring{D}^n$

$$\begin{tikzcd}
z_n \arrow[r, phantom, "\in"] \arrow[d, mapsto] & H_n(D^n, \partial D^n) \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H_n(D^n, D^n \setminus x) \arrow[d, "\cong"', "f_{1*}"] \arrow[dr, "\cong"', "f_{1*} = f_{1*}'"] & & H_n(D^n, \partial D^n) \arrow[ll, "\cong"'] \arrow[dd, "f_{1*}'"] \arrow[r, phantom, "\ni"] & z_n \arrow[dd, mapsto] \\
\operatorname{ldeg}(f) z_n \arrow[r, phantom, "\in"] & H_n(D^n, \partial D^n) \arrow[r, "\cong"] \arrow[drrr, equal] & H_n(D^n, D^n \setminus f_1(x)) & H_n(D^n, D^n \setminus \mathring{Z}_1) \arrow[l, "\cong"'] \arrow[dr, "\cong"] & & \\
 & & & & H_n(D^n, \partial D^n) \arrow[r, phantom, "\ni"] & \deg(f_1'
\end{tikzcd}$$

$f_1 \simeq f_1' : (D^n, \partial D^n) \longrightarrow (D^n, D^n \setminus \mathring{Z}_1)$

---

[*] This is the only step in which $\pi_{n-1}S^{n-1} \cong \mathbb{Z}$ was used.

Topology II, lecture 21 27.06.2018

So far we've been using disks $(D^n, \partial D^n)$.

More generally we may use any abstract disc $(B, \partial B)$; an abstract disc is understood to be a space $B \cong D^n$; any such B has $\partial B \cong \partial D^n$.

Then we have $g: (B, \partial B, *) \longrightarrow (X, A, x_0)$ and $f_i: (C, \partial C) \longrightarrow (B, \partial B)$ where $(C, \partial C)$ and $(B, \partial B)$ are oriented — thus ldeg f_i makes sense.

Eg. we can use $(\Delta^n, \partial\Delta^n)$ in HAT with canonical orientation Id_{Δ^n}.

Lemma 4. $g: (D^n, \partial D^n, *) \longrightarrow (X, A, x_0)$, $f_i: D^n \longrightarrow D^n$, $f_i|_{\mathring{D}^n}: \mathring{D}^n \xrightarrow{\cong} f_i(\mathring{D}^n)$, $f_i(\mathring{D}^n) \cap f_j(\mathring{D}^n) = \emptyset$ $\forall i \neq j$, $g\left(D^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{D}^n)\right) \subseteq A$. (The usual setup)

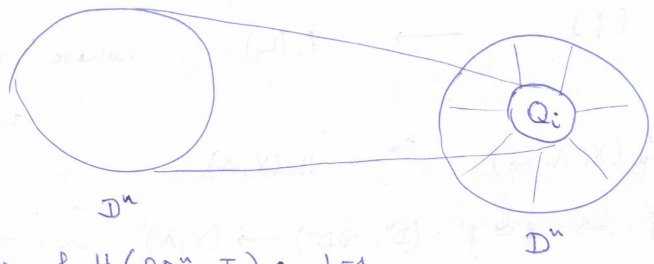
Given $U_i \subseteq f_i(\mathring{D}^n)$ open there exist replacements f_i' for f_i s.t. $f_i'(D^n) \subseteq U_i$ (here replacement means that g can also be replaced by g' and $\gamma_{i*}[g \circ f_i] = \gamma_{i*}'[g' \circ f_i']$.)

Pf: Lemma 3 $\Rightarrow$ wlog wma $0 \in Q_i \subseteq D^n$ concentric s.t. $f_i(Q_i) \subseteq U_i \subseteq f_i(D^n)$. $\exists\, g' \simeq g$ (as pairs) s.t. $g'\left(D^n \setminus \bigcup_{i=1}^{k} f_i(\mathring{Q}_i)\right) \subseteq A$, $\gamma_*[g' \circ f_i] = \gamma_*[g \circ f_i]$.

Consider homotopies $H_i: D^n \times I \longrightarrow D^n$, $H_i(-, 0) = id$, $H_i(-, 1): D^n \xrightarrow{\cong} Q_i \subseteq D^n$, $H_i(\partial D^n \times I) \subseteq D^n \setminus \mathring{Q}_i$

Such homotopies exist.

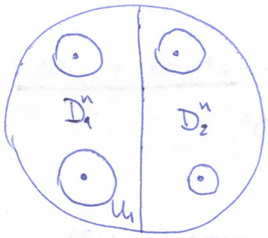
Consider $f_i \circ H: D^n \times I \longrightarrow D^n$.



Let $f_i' := f_i \circ H(-, 1)$. Then $f_i H(\partial D^n \times I) \subseteq g'^{-1}(A)$. The conditions of Lemma 3 are satisfied $\Rightarrow$ $\gamma_{i*}[g' \circ f_i] = \gamma_{i*}'[g' \circ f_i']$.

Recall that we stated $\pi_{n-1}S^{n-1} \cong \mathbb{Z}$ $\overset{(\#)}{\text{implies}}$ HAT in dim n. $(n \geq 2)$.
We do induction on k; the case $k = 1$ was covered in lecture 20.
Suppose (#) holds for $< k$; we now show it for k.

Choose points $u_i \in f_i(\mathring{D}^n)$, then divide D^n into two parts D_1^n, D_2^n



in such a way that both discs contain at least one of the U_i in their interior.

Find $U_i \subseteq f_i(\mathring{D}^n)$ s.t. $u_i \in U_i$ and each U_i is contained in the interior of one of the half-discs.

Now wlog wma $f_i(D^n) \subseteq D_1$ or $\subseteq D_2$ because otherwise we can apply the previous Lemma 4 to replace the f_is. Choose orientations of D_1 and D_2 in such a way that $D_1 \hookrightarrow D^n$, $D_2 \hookrightarrow D^n$ have local degree 1. Then $[g] = [g_1] + [g_2]$ (maybe in $\tilde{\pi}_n$?)

Induction assumption $\Rightarrow$ $= \sum_{\substack{i \\ u_i \in D_1}} \mathrm{ldeg}(f_i)\, \gamma_{1*}[g_1 \circ f_i] + \sum_{\substack{i \\ u_i \in D_2}} \mathrm{ldeg}(f_i)\, \gamma_{2*}[g_2 \circ f_i]$

Note that both sums have $< k$ terms. □

Now we will actually start proving Hurewicz. (For reference, see Bredon; the way this is treated in Waldhausen uses heavy machinery, that is, geometric realisation.)

We have seen: $\pi_{n-1} S^{n-1}$ $\Rightarrow$ HAT in dim n.

We will show that HAT in dim n $\Rightarrow$ relative Hurewicz in dim n.

Note that (relative) Hurewicz in dim n $\Rightarrow$ $\pi_n S^n \cong H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$.

We do already know that $\pi_1 S^1 \cong \mathbb{Z}$.

Let $h_n : \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$

$[f] \longmapsto f_*(z_n)$ where $z_n \in H_n(D^n, \partial D^n)$

$\wr\wr$

$z_n = [id] \in H_n(\Delta^n, \partial \Delta^n)$

Factor over $\tilde{\pi}_n(X, A, x_0) \xrightarrow{h_n} H_n(X, A)$.

$\gamma_*[f] = [f'] \Rightarrow f \simeq f' : (D^n, \partial D^n) \longrightarrow (X, A) \Rightarrow f_* = f'_*$

for $[\gamma] \in \pi_1(A, x_0)$.

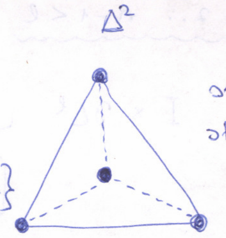
We reformulate the statement:

Thm. (Hurewicz) Let X, A be path-connected spaces s.t. $\pi_k(X, A, x_0) = 0$ for $k \leq n-1$ where $n \geq 2$. Then $\tilde{H}_k(X, A) = 0$ $\forall k \leq n-1$ and the map

$h_n : \tilde{\pi}_n(X, A, x_0) \longrightarrow H_n(X, A)$ is an isomorphism.

Topology II, lecture 21

Consider Δ^k with n^{th} skeleton $sk_n \Delta^k$.



$sk_0 \Delta^2 = 4$ pts

$sk_1 \Delta^2 = \partial\Delta^2$

$C_*^{(m)}(X,A) \subseteq C_*(X,A)$ defined as $\underline{C_*^{(m)}(X,A)} = \{\sigma \in C_*(X,A) \mid \sigma(sk_m \Delta^k) \subseteq A\}$

Prop. If X, A are path-connected and $\pi_k(X,A,x_0) = 0$ for $k<n$ then

$$C_*^{(n-1)}(X,A) \hookrightarrow C_*(X,A)$$

is a chain homotopy equivalence.

Pf: For any singular l-simplex $\sigma: \Delta^l \longrightarrow X$ we will construct maps

$$p(\sigma): I \times \Delta^l \longrightarrow X$$

with the properties
(1) $p(\sigma)(0,-) = \sigma$
(2) $p(\sigma)(1,-) \in C_l^{(n-1)}(X,A)$
(3) $p(\sigma)(t,-) = \sigma$ if $\sigma \in C_l^{(n-1)}(X,A)$
(4) $p(\sigma)(1 \times d_i) = p(\sigma \circ d_i)$ $\forall 0 \leq i \leq l$.

If $\sigma \in C_l^{(n-1)}(X,A)$ let $p(\sigma)(t,z) := \sigma(z)$.

Assume $\sigma \notin C_l^{(n-1)}(X,A)$. We do induction on l.

$l=0$:
$$\begin{tikzcd} 0 \in \Delta^0 \arrow[r, "\sigma"] \arrow[d, mapsto] \arrow[d, hook, shift left=3] & X \\ 0 \in I \arrow[ur, "p(\sigma)"'] & \end{tikzcd}$$
$\sigma(\Delta^0) \notin A$. There is a path connecting $\sigma(\Delta^0)$ to a point in A.

$p(\sigma)(1) \in A$

$l>0$, assume the assumption is already shown for $\dim < l$.

$\sigma: \Delta^l \longrightarrow X$.

Case 1: $l \leq n-1$ $\quad sk_{n-1} \Delta^l = \Delta^l$.

$$(I \times \partial\Delta^l \cup 0 \times \Delta^l,\ 1 \times \partial\Delta^l) \xrightarrow{p(\partial\sigma) \cup \sigma} (X,A)$$

$p(\sigma \circ d_i)$ are already defined: $\partial\Delta^l = \bigcup_{i=0}^{l} d_i \Delta^{l-1}$

$\exists\, I \times \partial\Delta^l \xrightarrow{p(\partial\sigma)} X$ s.t. $p(\partial\sigma)\big|_{I \times d_i \Delta^l} = p(\sigma \circ d_i)$

$$\begin{tikzcd} (I \times \partial\Delta^l \cup 0 \times \Delta^l,\ 1 \times \partial\Delta^l) \arrow[r, "p(\partial\sigma) \cup \sigma"] \arrow[d, hook] & (X,A) \\ (I \times \Delta^l,\ 1 \times \Delta^l) \arrow[ur, dashed, "\exists\, p(\sigma)"'] & \end{tikzcd}$$
(choose one)

This is because $\pi_l(X,A,x_0) = 0$.

Case 2.: $n-1 < l$: $\quad sk_{n-1}\Delta^l = sk_{n-1}\partial\Delta^l$

$$\begin{tikzcd}
I \times \partial\Delta^l \cup 0 \times \Delta^l \arrow[r, "p(\partial\sigma)\cup\sigma"] \arrow[d, hook] & (X, A) \\
I \times \Delta^l \arrow[ur, dashed, "\exists p(\sigma)"'] &
\end{tikzcd}$$

choose one using HEP

(This case is easier because the lower dimensional cases are automatically taken care of.)

Now we define the homology cross product:

$$\begin{tikzcd}
C_l(X) \times C_k(Y) \arrow[r] & C_{l+k}(X \times Y) \\
\overset{\sigma}{\Delta^l \to X} \quad \overset{\tau}{\Delta^k \to Y} \arrow[r, mapsto] & \underline{\sigma \times \tau}
\end{tikzcd}$$

$$\Delta^l \times \Delta^k \to X \times Y$$
$$\Delta^l \times \Delta^k \cong \Delta^{l+k}$$

Well, actually we don't define it here, just treat its existence as a black box. (It will be an exercise.)

This has the property $\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^l \sigma \times \partial\tau$.

Also the hlg cross product is in some way inverse to the cup product (Alexander — Whitney).

Recall that we want to show that $C_*^{(n-1)}(X,A) \xhookrightarrow{\text{incl}} C_*(X,A)$ is a chain homotopy equivalence.

$$\begin{aligned}
\varphi : C_l(X,A) &\longrightarrow C_l^{(n-1)}(X,A) \\
\sigma &\longmapsto p(\sigma)(1, -)
\end{aligned}$$

recall: $p(\sigma) : I \times \Delta^l \longrightarrow X$

This gives a chain map since $p(\sigma \circ d_i) = p(\sigma)(\mathrm{id} \times d_i) = \varphi(\sigma \circ d_i) = d_i^*(\varphi(\sigma))$

$$\begin{tikzcd}
C_*^{(n-1)}(X,A) \arrow[r, hook, "\text{incl}"] \arrow[rr, bend right, "\mathrm{id}"'] & C_*(X,A) \arrow[r, "\varphi"] & C_*^{(n-1)}(X,A)
\end{tikzcd}$$

NTS $\text{incl} \circ \varphi \simeq \mathrm{id}$

Let $\iota_l \in C_*(\Delta^l)$ be the identity $\Delta^l \longrightarrow \Delta^l$.

$$\begin{aligned}
D : C_l(X,A) &\longrightarrow C_{l+1}(X,A) \\
\sigma &\longmapsto p(\sigma)_*(\iota_1 \times \iota_l)
\end{aligned}$$

For $\sigma : \Delta^l \longrightarrow X$, $p(\sigma) : I \times \Delta^n \longrightarrow X$ we have the following:

Topology II, lecture 21

$$\begin{tikzcd}
 & C_{\ell+1}(I\times\Delta^\ell) \arrow[r, "p(\sigma)_*"] & C_\ell(X) \\
\iota_1\times\iota_\ell \arrow[rr, mapsto] & & p(\sigma)_*(\iota_1\times\iota_\ell) \\
(\iota_1,\iota_\ell)\in \arrow[u] & C_1(I)\times C_\ell(\Delta^\ell) \arrow[uu] &
\end{tikzcd}$$

$$\begin{aligned}
d\,D(\sigma) &= d\,p(\sigma)_*(\iota_1\times\iota_\ell) = p(\sigma)_*(d(\iota_1\times\iota_\ell)) = p(\sigma)_*(d\iota_1\times\iota_\ell - \iota_1\times d\iota_\ell) \\
&= p(\sigma)_*(0\times\iota_1 - 1\times\iota_1) - p(\sigma)_*\left(\sum_{i=0}^{\ell}(-1)^i\,\iota_1\times d_i\right) \\
D(d\sigma) &= \sum_{i=0}^{\ell}(-1)^i\,D(\sigma\circ d_i) = \sum_{i=0}^{\ell}(-1)^i\,p(\sigma\circ d_i)_*(\iota_1\times\iota_{\ell-1}) \\
&= \sum_{i=0}^{\ell}(-1)^i\,p(\sigma)_*(1\times d_i)_*(\iota_1\times\iota_{\ell-1}) = \sum_{i=0}^{\ell}(-1)^i\,p(\sigma)_*(\iota_1\times d_i)
\end{aligned}$$

$$0\times\iota_\ell\in C_\ell(0\times\Delta^\ell) = C_\ell(\Delta^\ell)$$

$$d\,D(\sigma) + D(d\sigma) = p(\sigma)_*(0\times\iota_\ell - 1\times\iota) = \mathrm{id} - \varphi$$

<u>Cor.</u> $H_k(X,A)=0 \quad \forall k\le n-1$

Pf: Indeed, $C_*^{(n-1)}(X,A)$ is ch. htp. eq. to $C_*(X,A)$. Hence $H_k(C_*^{(n-1)}(X,A))\cong H_k(X,A)$. $\forall k\le n-1$. For $\sigma\in C_k^{(n-1)}(X,A)$: $\sigma(\mathrm{sk}_{n-1}\Delta^k)\subseteq A$, $\mathrm{sk}_{n-1}\Delta^k=\Delta^k \Rightarrow C_k^{(n-1)}(X,A)=0$

$\Rightarrow H_k(X,A)=0 \quad \forall k\le n-1$.

Note that up until now HAT was not used.

Also note that

$$h_n:\ \tilde{\pi}_n(X,A,x_0) \longrightarrow H_n(C_*^{(n-1)}(X,A)) \cong H_n(X,A)$$

$$[f:(\Delta^n,\partial\Delta^n)\to(X,A)] \longmapsto [f]$$

$$\mathrm{sk}_{n-1}\Delta^n = \partial\Delta^n,\quad f(\partial\Delta^n)\subseteq A$$

So it suffices to define the inverse

$$\psi_n:\ H_n(C_*^{(n-1)}(X,A)) \longrightarrow \tilde{\pi}_n(X,A).$$

We first do this on the chain level:

$$C_*^{(n-1)}(X,A) \longrightarrow \tilde{\pi}_n(X,A)$$

$$\underset{(\Delta^n,\partial\Delta^n)\to(X,A)}{f} \longmapsto \gamma_*[f] \qquad \gamma:\text{ a path from } f(*) \text{ to } x_0,\ D^n\cong\Delta^n$$

Note that $C_{n-1}^{(n-1)}(X,A)=0$.

To show that this descends to homology we need to show $\psi_n(\partial g) = 0$.

Then $\psi_n : H_n(C_n^{(n-1)}(X,A)) \longrightarrow \tilde{\pi}_n(X,A,x_0)$ will obviously be the inverse of h_n.

HAT in dim $n \Rightarrow \psi_n(\partial g) = 0$:

$g : \Delta^{n+1} \longrightarrow X, \quad g(sk_{n-1}\Delta^{n+1}) \subseteq A$

$$\psi_n(\partial g) = \psi_n\left(\sum (-1)^i g \circ d_i\right) = \sum_{i=0}^{n+1} (-1)^i \gamma_{i*}[g \circ d_i] = 0$$

$$\partial \Delta^{n+1} = \bigcup_{i=0}^{n+1} d_i \Delta^n$$

"Wait for 5 minutes, I haven't finished this proof. DON'T run away!"

Define a disc $B := d_0\Delta^n \cup d_1\Delta^n \cup d_2\Delta^n \cup \ldots \cup d_{n+1}\Delta^n$

$$\begin{tikzcd}
(B, \partial B) \arrow[r] \arrow[rrr, bend right, "\bar{g}"'] & (B, sk_{n-1}B) \arrow[r, "\text{quotient}", "\text{``more gluing''}"'] & (\partial\Delta^{n+1}, sk_{n-1}\Delta^{n+1}) \arrow[r, "g|_{\partial\Delta^{n+1}}"] & (X, A, x_0)
\end{tikzcd}$$

Img of ∂B in $sk_{n-1}\Delta^{n+1}$ is contractible (combinatorial exercise, pf. in Waldhausen).

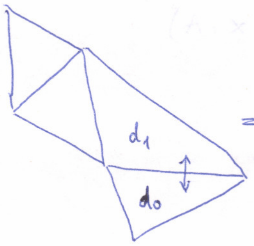
Up to homotopy, $\bar{g}$ is given by

$$\begin{tikzcd}
(B, \partial B) \arrow[r] & (\partial\Delta^{n+1}, *) \arrow[r, "g|_{\partial\Delta^{n+1}}"] \arrow[d] & (X, A) \\
 & (*,*) \simeq (\Delta^{n+1}, *) \arrow[ur] &
\end{tikzcd}$$

$\Rightarrow [\bar{g}] = 0$ in $\tilde{\pi}_n(X,A,x_0)$.

All the d_is are homeomorphic inclusions into B.

$$\text{HAT} \Rightarrow 0 = [\bar{g}] = \sum \underbrace{\operatorname{ldeg}(d_i)} \gamma_{i*} \underbrace{[\bar{g} \circ d_i]}_{= [g \circ d_i]} = \sum (-1)^i \gamma_{i*}[g \circ d_i]$$



$\Rightarrow \operatorname{ldeg}(d_i) = -\operatorname{ldeg}(d_{i+1})$

This finishes the pf.

"I think this is the best possible proof of Hurewicz theorem you will find in the literature." □